3.9) $(A, B)_W = tr(A^T W B)$, $W = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

a) Pruebo los axiomas de PI:

1) $(x+y, z) = tr((x+y)^T . W . z) = tr((x^T + y^T) W z) \longrightarrow$

$\rightarrow = tr(x^T W z) + tr(y^T W z) = (x, z) + (y, z)$ ✓

2) $(\lambda x, y) = tr((\lambda x)^T . W . y) = tr(\lambda . x^T . W . y) = \lambda . tr(x^T . W . y) = \lambda . (x, y)$ ✓

③ $\overline{(x,y)} = tr(x^T W B) = tr(B^T W x) = \underline{(y,x)}$ (son iguales) ✓

④ $\overline{(x,x)} = tr(x^T W x) = tr\left(\begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) \longrightarrow$

$\longrightarrow = tr\left(\begin{bmatrix} x_{11}+x_{21} & x_{11}+2x_{21} \\ x_{12}+x_{22} & x_{12}+2x_{22} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) =$

$\cancel{= tr\left(\left[x_{11}^2 + x_{21}x_{11} + x_{11}x_{21} + 2x_{21}^2 \right.\right.}$

$= tr\left(\begin{bmatrix} (x_{11}+x_{21})x_{11} + (x_{11}+2x_{21}) \cdot x_{21} & \cdots \\ \cdots & (x_{12}+x_{22})x_{12} + (x_{12}+2x_{22})x_{22} \end{bmatrix}\right) =$

$= tr\left(\begin{bmatrix} x_{11}^2 + 2x_{21}x_{11} + 2x_{21}^2 & \cdots \\ \cdots & x_{12}^2 + 2x_{22}x_{12} + 2x_{22}^2 \end{bmatrix}\right) =$

$= x_{11}^2 + 2x_{21}x_{11} + 2x_{21}^2 + x_{12}^2 + 2x_{22}x_{12} + 2x_{22}^2 =$

$= \left[(x_{11}+x_{21})^2 + x_{21}^2\right] + \left[(x_{12}+x_{22})^2 + x_{22}^2\right] \underline{\geq 0}$ ✓

Cumple todos los axiomas → define un PI en $\mathbb{R}^{2\times 2}$ ✓

b) $S = \text{gem}\left\{ \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{v_2} \right\}$ Son LI, es una base.

Dim(S) = 2

Busco base ortogonal de S, $B = \{w_1, w_2\}$

$w_1 = v_1$

$w_2 = v_2 - \dfrac{(v_2, v_1)}{\|v_1\|^2} \cdot v_1$

$\rightarrow w_1 = v_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ✓

$$w_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \frac{\overbrace{\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)}^{\mathrm{I}}}{\underbrace{\left\|\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right\|^2}_{\mathrm{II}}} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \Delta$$

$\mathrm{II} \rightarrow = (v_1, v_1)$

Uso PI dado:

$\mathrm{I} \rightarrow (v_2, v_1) = tr(v_2^T W . v_1) = tr\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) \longrightarrow$

$\rightarrow = tr\left(\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = tr\left(\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\right) = 2 + 3 = \boxed{5}$

$\mathrm{II} \rightarrow (v_1, v_1) = tr(v_1^T W . v_1) = tr\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) \longrightarrow$

$= tr\left(\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = tr\left(\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}\right) = 1 + 5 = \boxed{6}$

Por lo tanto:

(Δ) $\rightarrow w_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \frac{5}{6} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\rightarrow w_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 5/6 & 5/6 \\ 0 & 5/6 \end{bmatrix} \rightarrow w_2 = \begin{bmatrix} 1/6 & -5/6 \\ 1 & 1/6 \end{bmatrix}$

Entonces la base buscada es $\boxed{\beta = \{w_1, w_2\} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1/6 & -5/6 \\ 1 & 1/6 \end{bmatrix} \right\}}$

Ahora, $S^{\perp} = \{ v \in \mathbb{R}^{2\times 2} : (v_1, v) = 0 \wedge (v_2, v) = 0 \}$

$\rightarrow (v_1, v) = 0 \rightarrow tr(v_1^T W . v) = 0 \rightarrow tr\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = 0 \longrightarrow$

$\rightarrow tr\left(\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = 0 \rightarrow tr\left(\begin{bmatrix} a+c & \cdots \\ \cdots & 2b+3d \end{bmatrix} \right) = 0 \rightarrow \boxed{a+c+2b+3d = 0}$ (I)

1ra condición.

$(v_2, v) = 0 \rightarrow tr(v_2^T W . v) = 0 \rightarrow tr\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = 0 \longrightarrow$

$\rightarrow tr\left(\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = 0 \rightarrow tr\left(\begin{bmatrix} 2a+3c & \cdots \\ \cdots & b+2d \end{bmatrix} \right) = 0 \longrightarrow$

$\rightarrow \boxed{2a+3c+b+2d = 0}$ (II)

2da condición

$\rightarrow S^{\perp} = \{ v \in \mathbb{R}^{2\times 2} : \underbrace{a+2b+c+3d=0}_{\mathrm{I}} \wedge \underbrace{2a+b+3c+2d=0}_{\mathrm{II}} \}$

Buscar ahora una base:

de I $\rightarrow a = -2b - c - 3d \rightarrow a = -\frac{2}{3}c + \frac{8}{3}d - c - 3d \rightarrow a = -\frac{5}{3}c - \frac{d}{3}$

en II $\rightarrow -4b - 2c - 6d + b + 3c + 2d = 0 \rightarrow -3b + c - 4d = 0 \rightarrow b = \frac{c}{3} - \frac{4}{3}d$

Por lo tanto:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -\frac{5}{3}c - \frac{d}{3} & \frac{c}{3} - \frac{4}{3}d \\ c & d \end{bmatrix} = c \cdot \begin{bmatrix} -\frac{5}{3} & \frac{1}{3} \\ 1 & 0 \end{bmatrix} + d \cdot \begin{bmatrix} -\frac{1}{3} & -\frac{4}{3} \\ 0 & 1 \end{bmatrix}$$

Por lo tanto al ser LI, una base de $S^{\perp}$ es:

$$\boxed{B = \left\{ \begin{bmatrix} -\frac{5}{3} & \frac{1}{3} \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} & -\frac{4}{3} \\ 0 & 1 \end{bmatrix} \right\}}$$

c) $A = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

$$B_S = \left\{ \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 1/6 & -5/6 \\ 1 & 1/6 \end{bmatrix}}_{v_2} \right\}$$

$$B_{S^\perp} = \left\{ \underbrace{\begin{bmatrix} -5/3 & 1/3 \\ 1 & 0 \end{bmatrix}}_{v_3}, \underbrace{\begin{bmatrix} -1/3 & -4/3 \\ 0 & 1 \end{bmatrix}}_{v_4} \right\}$$

Junto las dos bases:

$B_S \cup B_{S^\perp} = \{v_1, v_2, v_3, v_4\}$ ~~es base de~~

veo si son LI

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1/6 & -5/6 & 1 & 1/6 \\ -5/3 & 1/3 & 1 & 0 \\ -1/3 & -4/3 & 0 & 1 \end{pmatrix} \begin{array}{l} \\ F_2 \to \frac{1}{6}F_1 - F_2 \\ F_3 \to \frac{5}{3}F_1 + F_3 \\ F_4 \to \frac{1}{3}F_1 + F_4 \end{array} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 1 & 5/3 \\ 0 & -1 & 0 & 4/3 \end{pmatrix} \begin{array}{l} \\ \\ F_3 \to 2F_2 - F_3 \\ F_4 \to F_2 + F_4 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -3 & -5/3 \\ 0 & 0 & -1 & 4/3 \end{pmatrix} F_4 \to F_3 - 3F_4 \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -3 & -5/3 \\ 0 & 0 & 0 & -17/3 \end{pmatrix}$$

No se canceló ninguna fila → son LI.

Entonces $\{v_1, v_2, v_3, v_4\}$ → base de $\mathbb{R}^{2\times 2}$

$$\underbrace{\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}}_{A} = \underbrace{\alpha \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \beta \begin{bmatrix} 1/6 & -5/6 \\ 1 & 1/6 \end{bmatrix}}_{A_S} + \underbrace{\gamma \begin{bmatrix} -5/3 & 1/3 \\ 1 & 0 \end{bmatrix} + \theta \begin{bmatrix} -1/3 & -4/3 \\ 0 & 1 \end{bmatrix}}_{A_{S^\perp}}$$

Ec.:

$$
\begin{cases}
\alpha + \frac{\beta}{6} - \frac{5}{3}\gamma - \frac{\theta}{3} = 2 \\
\alpha - \frac{5}{6}\beta + \frac{\gamma}{3} - \frac{4}{3}\theta = 1 \\
\beta + \gamma = -3 \\
\alpha + \frac{\beta}{6} + \theta = 2
\end{cases}
$$

$$
\left(\begin{array}{cccc|c}
1 & \frac{1}{6} & -\frac{5}{3} & -\frac{1}{3} & 2 \\
1 & -\frac{5}{6} & \frac{1}{3} & -\frac{4}{3} & 1 \\
0 & 1 & 1 & 0 & -3 \\
1 & \frac{1}{6} & 0 & 1 & 2
\end{array}\right)
\begin{array}{l} \\ F2 \to F1 - F2 \\ \\ F4 \to F1 - F4 \end{array}
\left(\begin{array}{cccc|c}
1 & 1/6 & -5/3 & -1/3 & 2 \\
0 & 1 & -2 & 1 & 1 \\
0 & 1 & 1 & 0 & -3 \\
0 & 0 & -5/3 & -4/3 & 0
\end{array}\right) F3 \to F2 - F3
$$

$$
\left(\begin{array}{cccc|c}
1 & 1/6 & -5/3 & -1/3 & 2 \\
0 & 1 & -2 & 1 & 1 \\
0 & 0 & -3 & 1 & 4 \\
0 & 0 & -5/3 & -4/3 & 0
\end{array}\right) F4 \to \frac{5}{3}F3 - 3F4
\qquad
\left(\begin{array}{cccc|c}
1 & 1/6 & -5/3 & -1/3 & 2 \\
0 & 1 & -2 & 1 & 1 \\
0 & 0 & -3 & 1 & 4 \\
0 & 0 & 0 & 17/3 & 20/3
\end{array}\right)
$$

$$
\to
\begin{cases}
\alpha + \frac{\beta}{6} - \frac{5}{3}\gamma - \frac{\theta}{3} = 2 \to \alpha = \frac{35}{102} - \frac{80}{51} + \frac{20}{51} + 2 \to \boxed{\alpha = \frac{169}{102}} \to \boxed{\alpha = \frac{7}{6}} \\
\beta - 2\gamma + \theta = 1 \to \beta = 1 + 2\cdot\left(\frac{-16}{17}\right) - \frac{20}{17} \to \boxed{\beta = -\frac{35}{17}} \\
-3\gamma + \theta = 4 \to \gamma = \left(4 - \frac{20}{17}\right)\cdot\frac{-1}{3} \to \gamma = \frac{\overset{-16}{\cancel{-48}}}{\underset{17}{\cancel{51}}} \to \boxed{\frac{-16}{17} = \gamma} \\
\frac{17}{3}\theta = \frac{20}{3} \to \boxed{\theta = \frac{20}{17}}
\end{cases}
$$

$$
\underbrace{\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}}_{A} = \underbrace{\frac{7}{6}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \frac{35}{17}\begin{bmatrix} 1/6 & -5/6 \\ 1 & 1/6 \end{bmatrix}}_{As} \underbrace{- \frac{16}{17}\begin{bmatrix} -5/3 & 1/3 \\ 1 & 0 \end{bmatrix} + \frac{20}{17}\begin{bmatrix} -1/3 & -4/3 \\ 0 & 1 \end{bmatrix}}_{Ast}
$$